Counting the number of spanning trees in a graph - A spectral approach

April 29th, 2010

In class we came across a metric that required us to compute the number of spanning trees of a graph. We provide here some discussion on how this is done (efficiently) using spectral graph theory (essentially graph theory + linear algebra). Some of the notation and definitions are borrowed from Wikipedia's relevant articles. We start with some basic definitions that we will need.

Definition (Spanning Tree)**.** *Given a connected undirected graph $G = (V, E)$, a spanning tree is a subgraph $H \subseteq G$ such that H is a tree over the entire vertex set of G.*

Definition (Graph Laplacian)**.** *Given an undirected graph with an ordering of its vertices $(v_1, v_2, \ldots, v_n)$, the* Laplacian matrix *$L(G)$ is defined to be a $n \times n$ matrix with the following entries:*

$$\ell_{i,j} = \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

That is, the diagonal elements have values equal to the degree of the corresponding vertices, and the off-diagonal elements are -1 if an edge connects the two vertices, and 0 otherwise.

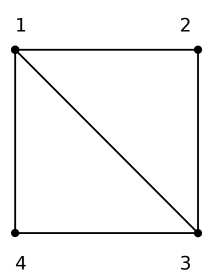
The Laplacian matrix of a graph has many interesting properties. In particular, one can show that for a connected graph, it's Laplacian matrix has $n - 1$ non-zero eigen values. This is used in the following theorem.

Theorem (Kirchhoff's Matrix Tree Theorem)**.** *For a given undirected connected graph G with n vertices, let $\lambda_1, \ldots, \lambda_{n-1}$ be the non-zero eigenvalues of $L(G)$. Then, the number of distinct spanning trees of G is equal to*

$$t(G) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i$$

Equivalently, $t(G)$ is equal to the absolute value of any cofactor of the Laplacian matrix of G.

As an example, consider two examples, show in Figure 1 and Figure 2.

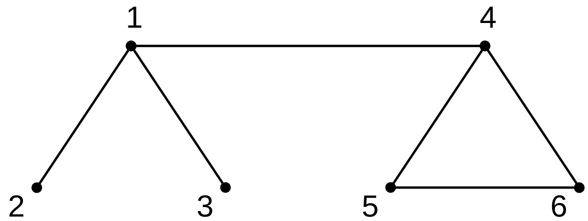


$$L(G) = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \qquad \begin{aligned} \lambda_1 &= 4 \\ \lambda_2 &= 4 \\ \lambda_3 &= 2 \\ \lambda_4 &= 0 \end{aligned} \qquad t(G) = \frac{1}{4}(4)(4)(2) = 8$$

Figure 1: Example graph, with Laplacian matrix and eigenvalues. Numbers near each vertex indicate the chosen ordering. The total number of spanning trees can be seen to be 8 by inspection, which matches with Kirchhoff's theorem.

We will now provide some intuition as to why Kirchhoff's theorem is correct. From spectral graph theory, we know that the Laplacian matrix of a graph G can be decomposed into the product of the incidence matrix E with it's transpose:

$$L(G) = EE^T$$



$$L(G) = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

$$\begin{aligned} \lambda_1 &\approx 4.5646 \\ \lambda_2 &= 3 \\ \lambda_3 &= 1 \\ \lambda_4 &= 1 \\ \lambda_5 &\approx 0.4384 \\ \lambda_6 &= 0 \end{aligned} \qquad t(G) = \frac{1}{6}(4.5616)(3)(1)(1)(0.4384) = 3$$

Figure 2: 2nd example graph, with Laplacian matrix and eigenvalues. Numbers near each vertex indicate the chosen ordering. The total number of spanning trees can be seen to be 8 by inspection, which matches with Kirchhoff's theorem.

The incidence matrix E for a graph with n nodes and m edges is a $n \times m$ matrix which indicates which edges are incident on which nodes. We assume both the edges and nodes are given an ordering. Using the ordering of the nodes, we impose a direction on each edge such that the edge points from the lower-ordered vertex to the higher ordered vertex. The entries of the incidence matrix are defined as follows:

$$a_{i,j} = \begin{cases} 1 & \text{if edge } e_j \text{ points out from } v_i \\ -1 & \text{if edge } e_j \text{ points to } v_i \\ 0 & \text{otherwise} \end{cases}$$

$$E_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

Figure 3: Incidence matrices of the two example graphs using the same node orderings as before and for a fixed edge ordering. Incidence matrices resulting from other edge orderings would be column permutations of the display matrices.

The incidence matrices for the two example graphs are shown in Figure 3. We wish to talk about the minor $\det(M_{11})$ of the Laplacian matrix L, i.e. the determinant of the matrix resulting from the removing the first row and column from L. We know that $L = EE^T$, so by letting F be the matrix produced by removing the first row from E, we can relate F to M_{11} in a similar way: $M_{11} = FF^T$. We now utilize a theorem by Cauchy and Binet:

Theorem (Cauchy-Binet formula)**.** *Let A be a $m \times n$ matrix and B be an $n \times m$ matrix. We write $[n]$ for the set $\{1, 2, \ldots, n\}$ and $\binom{[n]}{m}$ for all m-size subsets of $[n]$. For any $S \in \binom{[n]}{m}$, $A_{[m],S}$ is the $m \times m$ matrix produced by taking from A only the columns that are indexed by S. Similar notation is used for B, except instead we select rows by index instead of columns. Then,*

$$\det(AB) = \sum_{S \in \binom{[n]}{m}} \det\left(A_{[m],S}\right) \det\left(B_{S,[m]}\right)$$

We will apply the Cauchy-Binet formula to the relationship $M_{11} = FF^T$. Thus, we have

$$\det(M_{11}) = \sum_S \det(F_S) \det\left(F_S^T\right) = \sum_S \det(F_S)^2$$

Note here that M_{11} is of size $(n-1) \times (n-1)$, so S is chosen over $n-1$ sized subsets of $\{2, 3, \ldots, m\}$, and thus S specifies $n-1$ columns of F that constitute F_S. We can think of this subset S as all possible choices of $n-1$ edges. As all trees have $n-1$ edges, each choice of S *could* map to a choice of a spanning tree. We'd like to decide which of these choices of S represent spanning trees, and which do not. Fortunately, there is a relationship between the determinant of F_S and exactly this. We claim (without proof) that $\det(F_S)$ is equal to -1 or 1 if and only if the edges specified by S induce a spanning tree, and the determinant is equal to 0 iff S does not. Thus, $\det(F_S)^2$ becomes an indicator variable for a choice of S which takes the value 1 if S induces a spanning tree, and 0 otherwise. The right-hand summation of the above equation therefore is a count of the total number of spanning trees, which (by the Cauchy-Binet formala) is equivalent to minor $\det(M_{11})$. Our argument is symmetric for other selected minors (say an arbitrary minor $\det(M_{ij})$.

Thus we have that the determinant of a minor of of $L(G)$ is equal to the number of spanning trees. There is a result (although I'm not sure where it is to reference it) that shows equivalence between the these minors and the alternative expression $\frac{1}{n}\prod_i \lambda_i$.